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COMPATIBILITY AND MUTUAL INTERACTION AMONG THE DIMENSIONLESS GROUPS PRESENT IN THE NAVIER–STOKES AND ENERGY DIFFERENTIAL EQUATIONS

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Abstract – A method is proposed for obtaining a compatibility relationship among the Prandtl, Reynolds, Grashof and Eckert numbers, present in the transport differential equations for heat and momentum. This method enables one to calculate consistent multipliers for all terms in the aforesaid equations without reference to any physical system. The analysis reveals that natural convection and viscous dissipation cannot simultaneously affect solutions to transport equations with an equal and high intensity.

NOMENCLATURE

C, C_1, C_2, C_3, some constants;
C_v, specific heat;
Ec, Eckert number;
g, acceleration due to gravity;
Gr, Grashof number;
$\mathbf{k}$, versor opposed to gravity;
L, a characteristic length;
Pr, Prandtl number;
Re, Reynolds number;
t, dimensionless time;
T, dimensionless temperature;
$\mathbf{v}$, dimensionless velocity vector;
V, a characteristic velocity.

Greek letters

β, thermal expansion coefficient;
γ, a dimensionless parameter;
δ, a dimensionless parameter;
ΔT, a characteristic difference of temperature;
ε, a dimensionless parameter;
η, a dimensionless parameter;
κ, thermal diffusivity;
ν, kinematic viscosity;
ϕ, dimensionless dissipation function;
χ, a dimensionless parameter.

1. INTRODUCTION

IT IS WELL known that the dimensionless Navier–Stokes and energy differential equations for a Newtonian, incompressible, non-isothermal fluid, whose physical properties are supposed to be independent of temperature, except for the buoyancy term, depend on four dimensionless groups: Prandtl, Reynolds, Grashof and Eckert numbers. Their interaction, so complex in many cases, makes it difficult to predict the influence of each of them on solutions to transport differential equations. If a general study of this influence is carried out without reference to any prefigured physical system, the first problem of compatibility of the four numbers must be faced, since it will later be seen that if four random numbers are chosen, they may not individualize a meaningful physical situation.

The second problem should be to determine, for each dimensionless group, the amplitude and position of the range, within which values affecting solutions fall, bearing in mind that their mutual interactions must also be taken into account. These delimitations would make it possible to recognize *a priori* whether or not a certain term (inertia, buoyancy, dissipation, etc.) in the transport equations will be important in conditioning the shape of solutions.

Arguments in this paper endeavour to supply some explanations for the two problems outlined.

2. BASIC CONSIDERATIONS

The full dimensionless Navier–Stokes and energy differential equations for a Newtonian, incompressible, non-isothermal fluid can be written in Gibbs notation as follows:

$$\frac{\delta \mathbf{v}}{\delta t} = -(\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{Gr}{Re^2}T\mathbf{k} + \frac{1}{Re}\nabla^2\mathbf{v}$$
$$\frac{\delta T}{\delta t} = -(\mathbf{v}\cdot\nabla)T + \frac{1}{Pr\cdot Re}\nabla^2 T + \frac{Ec}{Re}\phi. \tag{1}$$

Physical properties are assumed to be independent of temperature except for the buoyancy term, in which the approximation of Boussinesq is used. The four numbers

$$Pr = \frac{\nu}{\kappa} = \gamma$$
$$Re = \frac{VL}{\nu} = \delta$$
$$Gr = \frac{g\beta L^3\,\Delta T}{\nu^2} = \varepsilon \tag{2}$$
$$Ec = \frac{V^2}{C_v\,\Delta T} = \eta$$

appear in the equations. They are now equated to $\gamma, \delta, \varepsilon, \eta$ for the sake of brevity and convenience, as will later become clear.

It is useful to note that all the quantities in (2) can be grouped as follows:

(1) physical properties of the fluid: C_v, β, κ, ν;
(2) geometry and operating conditions of the physical system: L, V, ΔT;
(3) dimensionless parameters: γ, δ, ε, η.

Fixing quite independently the four numbers $\gamma, \delta, \varepsilon, \eta$ can lead to:

(1) A quatern of incompatible physical properties, in that they are not simultaneously attributable to any existing fluid, if geometry and operating conditions of the system are supposed already fixed. In fact we obtain from (2):

$$C_v = \frac{V^2}{\Delta T} \cdot \frac{1}{\eta}$$

$$\beta = \frac{1}{g} \cdot \frac{V^2}{L \cdot \Delta T} \cdot \frac{\varepsilon}{\delta^2}$$

$$\kappa = L \cdot V \cdot \frac{1}{\gamma\delta}$$

$$\nu = L \cdot V \cdot \frac{1}{\delta}.$$

(2) Geometries and/or operating conditions which are again incompatible, in that they are not attributable to any physical system of interest (natural or man-made), if the working fluid is given. In fact the following is derived from (2):

$$L = \frac{1}{g} \cdot \frac{C_v}{\beta} \cdot \frac{\varepsilon\eta}{\delta^2} \tag{3}$$

$$V = g \cdot \frac{\beta\nu}{C_v} \cdot \frac{\delta^3}{\varepsilon\eta} \tag{4}$$

$$\Delta T = g^2 \cdot \frac{\beta^2\nu^2}{C_v^3} \cdot \frac{\delta^6}{\varepsilon^2\eta^3}. \tag{5}$$

γ does not appear in (3)–(5) because only the last three equations (2) are necessary and sufficient to get L, V, ΔT. It is here simply stressed that a problem of compatibility exists among the numbers γ, δ, ε, η. To cast light on their close connection without reference to any physical system, the employment of (5) as a link of $\delta, \varepsilon, \eta$ to ΔT and the properties of a fluid is proposed:

$$\eta^3 = \frac{g^2\beta^2\nu^2}{C_v^3\,\Delta T} \cdot \frac{\delta^6}{\varepsilon^2}. \tag{5'}$$

If a new dimensionless group is defined:

$$\chi = \frac{C_v^3\,\Delta T}{g^2\beta^2\nu^2}, \tag{6}$$

(5′) can be written:

$$\eta = \chi^{-1/3} \cdot \delta^2 \cdot \varepsilon^{-2/3}. \tag{7}$$

This is a compatibility relationship among δ, ε and η–χ which, as is shown by (6), depends on some physical properties of the working fluid and on ΔT. The range of the latter is generally more restricted than that of both L and V; particularly when physical properties are assumed to be approximately constant in a non-isothermal field of flow, ΔT must coherently be kept within a rather narrow range. As ΔT does not actually change so much as L and V, (5) can help us to grasp the general connection among δ, ε and η more easily than (3) and (4); for this reason it was preferred for the derivation of a compatibility relationship.

χ could obviously be expressed in terms of the already defined numbers: (5′) can be written

$$\chi = \frac{Re^6}{Gr^2 \cdot Ec^3}$$

but this expression would not give any immediate indication as to its dependence on the fluid and ΔT. So definition (6) is more meaningful. χ is generally a very large number for usual values of ΔT. It is computed for $\Delta T = 1°C$ in Table 1 and quoted for several fluids together with values of their pertinent physical properties. Of course, it depends on some thermodynamic coordinates, in particular on temperature.

If ε and η are the coordinates of a cartesian frame (a different couple among δ, ε, η could be chosen; the above choice seemed the most convenient, because more help comes from physical intuition and habit in fixing δ), plots of $\eta = \eta\ (\delta, \varepsilon)$, each labelled by an assigned value of δ, can be traced according to (7). As an example, this is done in Fig. 1, where three families of curves (7), corresponding to three values of χ, are drawn. As logarithmic scales are used for η and ε, functions $\eta = \eta\ (\delta, \varepsilon)$, $\delta = C$, are represented by straight lines. It is seen that, whatever δ is, η decreases for an increasing ε, η increases with δ, ε being constant. It must be noted, however, that the ratios $1/\delta$, $1/\gamma\delta$, ε/δ^2, η/δ are actually present as multipliers in the general transport equations (1). If $\delta, \varepsilon, \eta$ satisfy (7) for a given χ, a tern of consistent ratios $1/\delta$, ε/δ^2, η/δ, whose values might be correctly introduced in (1), is immediately computed and holds for all fluids and thermal states characterized by the same χ.

$1/\gamma\delta$ is calculated by the Prandtl number. Of course γ must be compatible with χ because we have from (6):

$$\gamma^2 = \frac{1}{\chi} \cdot \frac{C_v^3 \cdot \Delta T}{g^2\beta^2\kappa^2}$$

$$\gamma = \frac{1}{g}\left(\frac{\Delta T}{\chi}\right)^{1/2} \cdot \frac{C_v^{3/2}}{\beta\kappa}.$$

As different fluids can have the same χ (this can also be argued from Table 1 in a few cases), different values of γ are consistent with the same χ; if it is for two different fluids and the same ΔT:

$$\frac{C_{v1}^3\,\Delta T}{g^2\beta_1^2\nu_1^2} = \frac{C_{v2}^3\,\Delta T}{g^2\beta_2^2\nu_2^2},$$

then

$$\frac{C_{v1}^3\,\Delta T}{g^2\beta_1^2\kappa_1^2} \neq \frac{C_{v2}^3\,\Delta T}{g^2\beta_2^2\kappa_2^2}.$$

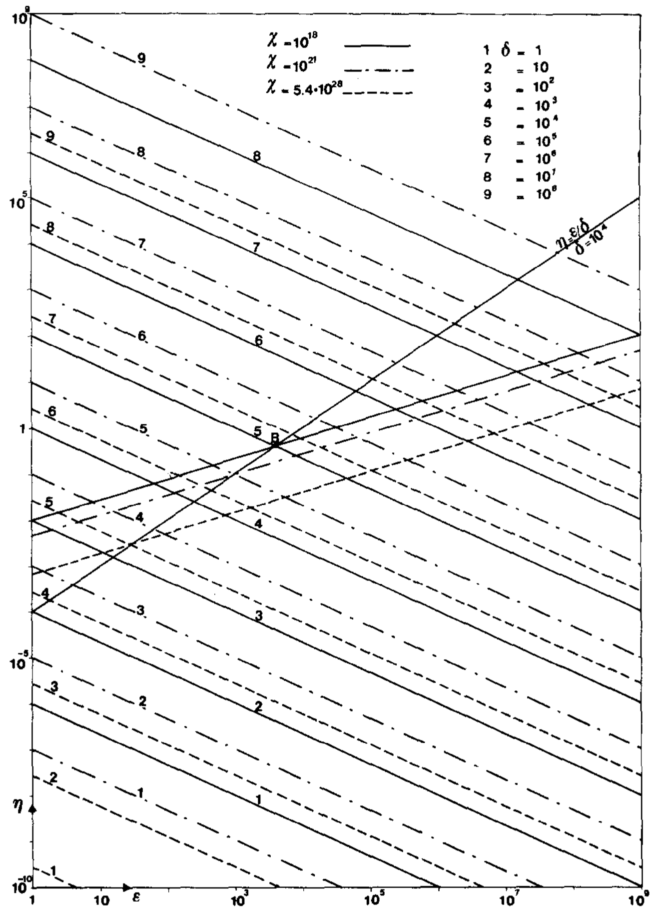

FIG. 1. A graphic representation of the compatibility relationship (7) for three values of χ and the related three loci (12) where $\varepsilon/\delta^2 = \eta/\delta$.

since generally

$$\frac{\nu_2}{\nu_1} \neq \frac{\kappa_2}{\kappa_1}.$$

It is concluded that γ cannot be extracted from χ. Calculation of $1/\gamma\delta$ demands an independent knowledge of γ.

Now that a criterion for compatibility is stated, it is useful to point out what the quaterns of consistent $1/\delta$, $1/\gamma\delta$, ε/δ^2, η/δ are and how they change when the representative point moves throughout the plane $\varepsilon\eta$. Thus analytical expressions for these ratios are obtained, valid when the point $\varepsilon\eta$ is moved along prescribed lines. As γ is constant for a given fluid at a

Table 1. Values of χ for several fluids and $\Delta T = 1°C$

Fluid	T (°C)	C_v(J kg^{-1}°C^{-1})	β (1/°C)	ν (m^2 s^{-1})	χ
Glycerine (sol. aq.)	−20	2100	0.28×10^{-3}	10^{-1}	1.23×10^{17}
Glycerine (sol. aq.)	0	2260	0.28×10^{-3}	8.31×10^{-3}	2.21×10^{19}
Lub. oil	0	1796	0.39×10^{-3}	4.28×10^{-3}	2.16×10^{19}
Lub. oil	20	1880	0.39×10^{-3}	0.9×10^{-3}	5.60×10^{20}
Air	0	716.4	3.66×10^{-3}	13.6×10^{-6}	1.53×10^{21}
Air	27	718	3.33×10^{-3}	15.7×10^{-6}	1.40×10^{21}
Air	200	737.3	2.11×10^{-3}	35.9×10^{-6}	7.23×10^{20}
Water	20	4182	0.18×10^{-3}	1.006×10^{-6}	2.32×10^{28}
Water	40	4178	0.18×10^{-3}	0.658×10^{-6}	5.40×10^{28}
Argon	0	313	3.66×10^{-3}	11.9×10^{-6}	1.68×10^{20}
Argon	200	311	2.11×10^{-3}	32.0×10^{-6}	6.86×10^{19}
Hydrogen	−23	9972	4.0×10^{-3}	80.6×10^{-6}	9.91×10^{22}
Hydrogen	177	10,283	2.22×10^{-3}	215.0×10^{-6}	4.95×10^{22}
Helium	−18	3132	3.92×10^{-3}	95.50×10^{-6}	2.28×10^{21}
Helium	177	3132	2.22×10^{-3}	269.0×10^{-6}	8.95×10^{20}
Mercury	0	140	0.18×10^{-3}	0.12×10^{-6}	5.71×10^{25}
Mercury	20	138	0.18×10^{-3}	0.115×10^{-6}	6.37×10^{25}
Mercury	100	137	0.18×10^{-3}	0.093×10^{-6}	9.60×10^{25}

reference temperature, the product $\gamma\delta$ is proportional to δ.

Along a straight line (Fig. 1) $\eta = C_1$, noting that (7) holds for every point of the $\varepsilon\eta$ plane, the set

$$\begin{cases} \eta = \chi^{-1/3} \cdot \delta^2 \cdot \varepsilon^{-2/3} \\ \eta = C_1 \end{cases}$$

gives:

$$\begin{aligned} \delta &= C_1^{1/2} \cdot \chi^{1/6} \cdot \varepsilon^{1/3} \\ 1/\gamma\delta &= C_1^{-1/2} \cdot \chi^{-1/6} \cdot \gamma^{-1} \cdot \varepsilon^{-1/3} \\ \varepsilon/\delta^2 &= C_1^{-1} \cdot \chi^{-1/3} \cdot \varepsilon^{1/3} \\ \eta/\delta &= C_1^{1/2} \cdot \chi^{-1/6} \cdot \varepsilon^{-1/3}. \end{aligned} \tag{8}$$

δ and ε/δ^2 grow with ε along straight lines parallel to the ε axis; η/δ, on the contrary, decreases.

Along any straight line $\varepsilon = C_2$ it is:

$$\begin{aligned} \delta &= C_2^{1/3} \cdot \chi^{1/6} \cdot \eta^{1/2} \\ 1/\gamma\delta &= C_2^{-1/3} \cdot \chi^{-1/6} \cdot \gamma^{-1} \cdot \eta^{-1/2} \\ \varepsilon/\delta^2 &= C_2^{1/3} \cdot \chi^{-1/3} \cdot \eta^{-1} \\ \eta/\delta &= C_2^{-1/3} \cdot \chi^{-1/6} \cdot \eta^{1/2}. \end{aligned} \tag{9}$$

δ and η/δ increase, ε/δ^2 decreases, for a growing η along straight lines parallel to the η axis. Table 2 is derived from calculation of the four relationships (9) for a wide range of η, four values of C_2, $\chi = 10^{18}$, $\gamma = 6 \times 10^4$; these two last numbers approximately fit physical properties of a lubricating oil at about $-15°C$ (Table 1). Results show that Fig. 1 can be divided into three horizontal strip fields: natural convection largely prevails in the lower strip, viscous dissipation in the upper one. Inside the intermediate strip, $10^{-2} < \eta < 10^2$, values of ε/δ^2 can be found of the same order as η/δ; this order is 10^{-4}, 10^{-5}, while it is $10^{-7} < 1/\delta < 10^{-2}$. Moving the representative point from left to right within every strip, natural convection always tends to become more important; the contrary occurs for viscous dissipation.

Along the descending straight lines of Fig. 1 we have:

$$\begin{aligned} \delta &= C_3 \\ 1/\gamma\delta &= C_3^{-1} \cdot \gamma^{-1} \\ \varepsilon/\delta^2 &= C_3^{-2} \cdot \varepsilon \\ \eta/\delta &= C_3 \cdot \chi^{-1/3} \cdot \varepsilon^{-2/3}. \end{aligned} \tag{10}$$

The last formula in (10) is nothing but (7) divided by δ. $1/\gamma\delta$ is not a function of ε; so, when δ and γ are fixed, the multiplier of the diffusive term computed from them is compatible with the infinitely numerous couples ε/δ^2 and η/δ. Figure 2 is a logarithmic graph of relationships (10) in the particular case of a lubricating oil. Right and left columns of stars quote $1/\delta$ and $1/\gamma\delta$ respectively. When δ is fixed, the four multipliers are immediately read in Fig. 2 for any ε and can be compared.

Results in Table 2 are related to a specified fluid; nevertheless they induce the doubt that it is impossible to have contemporaneously a strong viscous dissipation and a lively natural convection, since when ε/δ^2 is large η/δ is small; when the two ratios are of the same order, the latter is comparatively small. In order to clarify this point further, let us find the locus of the plane $\varepsilon\eta$ where

$$\varepsilon/\delta^2 = \eta/\delta \tag{11}$$

or

$$\eta = \varepsilon/\delta. \tag{11'}$$

(11′) represents a family of straight lines, each labelled by a value of δ. They are parallel in Fig. 1. The following equation of the locus is obtained from (7) and (11′):

$$\eta = \chi^{-1/9} \cdot \varepsilon^{4/9}; \tag{12}$$

Table 2. $1/\delta$, $1/\gamma\delta$, ε/δ^2, η/δ in terms of η along lines $\varepsilon = C_2$: $\chi = 10^{18}$; $\gamma = 6 \times 10^4$

η	$1/\delta$	$1/\gamma\delta$	ε/δ^2	η/δ
		$\varepsilon = C_2 = 1$		
10^8	10^{-7}	1.66×10^{-12}	10^{-14}	10
10^6	10^{-6}	1.66×10^{-11}	10^{-12}	1
10^4	10^{-5}	1.66×10^{-10}	10^{-10}	10^{-1}
10^2	10^{-4}	1.66×10^{-9}	10^{-8}	10^{-2}
1	10^{-3}	1.66×10^{-8}	10^{-6}	10^{-3}
10^{-2}	10^{-2}	1.66×10^{-7}	10^{-4}	10^{-4}
10^{-4}	10^{-1}	1.66×10^{-6}	10^{-2}	10^{-5}
10^{-6}	1	1.66×10^{-5}	1	10^{-6}
10^{-8}	10	1.66×10^{-4}	10^2	10^{-7}
		$\varepsilon = C_2 = 10^3$		
10^8	10^{-8}	1.66×10^{-13}	10^{-13}	1
10^6	10^{-7}	1.66×10^{-12}	10^{-11}	10^{-1}
10^4	10^{-6}	1.66×10^{-11}	10^{-9}	10^{-2}
10^2	10^{-5}	1.66×10^{-10}	10^{-7}	10^{-3}
1	10^{-4}	1.66×10^{-9}	10^{-5}	10^{-4}
10^{-2}	10^{-3}	1.66×10^{-8}	10^{-3}	10^{-5}
10^{-4}	10^{-2}	1.66×10^{-7}	10^{-1}	10^{-6}
10^{-6}	10^{-1}	1.66×10^{-6}	10	10^{-7}
10^{-8}	1	1.66×10^{-5}	10^3	10^{-8}
		$\varepsilon = C_2 = 10^6$		
10^8	10^{-9}	1.66×10^{-14}	10^{-12}	10^{-1}
10^6	10^{-8}	1.66×10^{-13}	10^{-10}	10^{-2}
10^4	10^{-7}	1.66×10^{-12}	10^{-8}	10^{-3}
10^2	10^{-6}	1.66×10^{-11}	10^{-6}	10^{-4}
1	10^{-5}	1.66×10^{-10}	10^{-4}	10^{-5}
10^{-2}	10^{-4}	1.66×10^{-9}	10^{-2}	10^{-6}
10^{-4}	10^{-3}	1.66×10^{-8}	1	10^{-7}
10^{-6}	10^{-2}	1.66×10^{-7}	10^2	10^{-8}
10^{-8}	10^{-1}	1.66×10^{-6}	10^4	10^{-9}
		$\varepsilon = C_2 = 10^9$		
10^8	10^{-10}	1.66×10^{-15}	10^{-11}	10^{-2}
10^6	10^{-9}	1.66×10^{-14}	10^{-9}	10^{-3}
10^4	10^{-8}	1.66×10^{-13}	10^{-7}	10^{-4}
10^2	10^{-7}	1.66×10^{-12}	10^{-5}	10^{-5}
1	10^{-6}	1.66×10^{-11}	10^{-3}	10^{-6}
10^{-2}	10^{-5}	1.66×10^{-10}	10^{-1}	10^{-7}
10^{-4}	10^{-4}	1.66×10^{-9}	10	10^{-8}
10^{-6}	10^{-3}	1.66×10^{-8}	10^3	10^{-9}
10^{-8}	10^{-2}	1.66×10^{-7}	10^5	10^{-10}

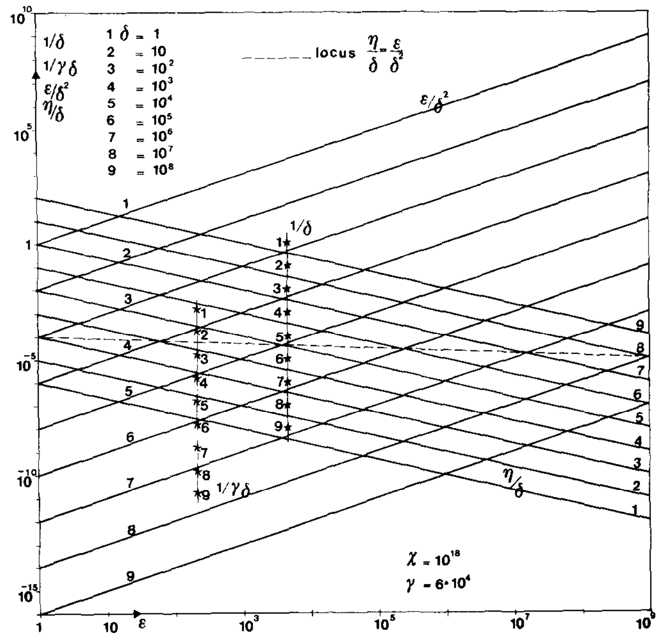

FIG. 2. Plots of ε/δ^2 and η/δ in terms of ε along lines (7) for $\chi = 10^{18}$ and several values of δ. Stars quote $1/\delta$. $1/\gamma\delta$ for a lubricating oil at $\sim -15°C$ is marked by the column of asterisks. The locus (12) for the same χ is also traced.

along (12) it is:

$$\delta = \chi^{1/9} \cdot \varepsilon^{5/9}$$
$$1/\gamma\delta = \chi^{-1/9} \cdot \gamma^{-1} \cdot \varepsilon^{-5/9} \qquad (13)$$
$$\varepsilon/\delta^2 = \eta/\delta = \chi^{-2/9} \cdot \varepsilon^{-1/9}.$$

Loci for $\chi = 10^{18}$, $\chi = 10^{21}$, $\chi = 5.4 \times 10^{28}$ (water at 40°C for example) are drawn in Fig. 1 as straight lines with a positive slope. Above every locus $\eta/\delta > \varepsilon/\delta^2$; below it the contrary holds. It must also be observed that every point on any straight line (11′) meets the condition (11), but only the intersection point between (7) and (11′) for the same δ (e.g. the point B in Fig. 1) meets (11) for the given χ.

η grows with ε more slowly than δ along (12), so η/δ, as is shown by the third relationship (13), decreases for a growing ε; the largest values are then found for the smaller ε, but they are at any rate small because of the magnitude of χ for existing fluids. Table 3 quotes η/δ for $\varepsilon = 1$ and various loci (12) whose χ values, although not strictly computed by the physical properties of well identified fluids, are within the range of Table 1. It is then confirmed that equal and relatively large values of ε/δ^2 and η/δ cannot coexist for any physical system; their equality implies smallness, except for $0 < \varepsilon < 1$, which is clearly not very interesting.

Table 3. η/δ for $\varepsilon = 1$ and several χ's on the loci (12)

χ	δ	$\varepsilon/\delta^2 = \eta/\delta$
10^{18}	10^2	10^{-4}
10^{21}	2.15×10^2	2.15×10^{-5}
10^{24}	4.64×10^2	4.64×10^{-6}
10^{29}	1.66×10^3	3.59×10^{-7}

As is apparent, all the formulae (7)–(13) depend on χ in a very heavy way since it is generally very large. When it is changed, say from χ_1 to χ, the curves (7) in Fig. 1 are shifted in the η direction by a quantity

$$\log \eta - \log \eta_1 = \Delta\eta = \log\left(\frac{\chi_1}{\chi}\right)^{1/3}.$$

If $\chi > \chi_1$, $\Delta\eta < 0$ and the curves (7) are shifted downwards by an increasing χ. The curves (12) are moved in the same way by an amount

$$\Delta\eta = \log\left(\frac{\chi_1}{\chi}\right)^{1/9}.$$

The change in χ directly acts upon descending straight lines η/δ in Fig. 2, whose ordinates are lowered when χ is raised. This is stated by the last relationship (10), the only one in this group which explicitly depends on χ. An increase of this number has then the direct effect of reducing the incidence of the viscous dissipation on solutions to (1); in this way χ can supply an immediate indication: the larger it is, the weaker this influence is.

Equation (6) shows that χ depends on several physical properties of the fluid, on g and ΔT. So stronger gravitational fields and weaker thermal fields yield a reinforcement of viscous dissipation effects.

A change in χ might imply a change in γ (see (6′)); an indirect effect on the diffusion term would result in this case. But the variation trend of $1/\gamma\delta$ cannot be known if the working fluid is not exactly specified. Equation (7) says that any two of the three numbers δ, ε, η can assume any value whatsoever and these are always consistent with any fluid; according to the viewpoint of this work, as the couple δ, ε is chosen, we can say that the effects of $1/\delta$ and ε/δ^2 on solutions to (1) are not conditioned by the nature of the fluid. When $\varepsilon < \delta$, natural convection should not exert as great an influence as viscous force, but both these effects are weakened by an increasing δ, as is shown in Fig. 2. The greater δ is, the greater ε must be for natural convection to affect solutions.

Equation (7) is introduced in (3) and (4) to eliminate their dependence on η and use is made of (6); the following expressions are derived:

$$L = (g \cdot \beta \cdot \Delta T)^{-1/3} \cdot \nu^{2/3} \cdot \varepsilon^{1/3} \quad (14)$$

$$V = (g \cdot \beta \cdot \Delta T)^{1/3} \cdot \nu^{1/3} \cdot \delta \cdot \varepsilon^{-1/3}. \quad (15)$$

L is a function only of ε and grows with it for a given fluid. Thus L is the same for any couple δ, η which, together with ε and χ, satisfies (7). Actually (14) is nothing but the expression of L extracted from Gr, but the way in which it is obtained here is deemed more significant. V decreases for an increasing ε, but also depends on δ. So one line $V = V\ (\delta, \varepsilon)$, $\delta = C$, is associated with each $\eta = \eta\ (\delta, \varepsilon)$ of Fig. 1 for the same δ.

With regard to the dependence of (14 and (15) on the physical properties, L and V do not depend on C_v. Besides, β does not change very much from one fluid to another and ΔT is in some way a fixed quantity. Therefore ν is the only property which substantially accounts for the peculiarity of the fluid in (14), (15). These functions are plotted on a logarithmic chart (Fig. 3) for two liquids with very different kinematic viscosities: water at 40°C and a lubricating oil at ~ -15°C. The combined observation of figures like 2 and 3 enables one to link dimensions and operating conditions of any physical system to the incidence of the various effects (viscous force, buoyancy, thermal diffusion, viscous dissipation). As an example, some natural systems which are often characterized by a large L and a small V, should more likely be the seat of a lively natural convection rather than strong dissipative phenomena; a well known result.

Arguments in this work construct a context which outlines how the interaction among the four dimensionless groups develops; the context can contribute to the acquisition of a first, rough orientation in assessing the influence of the various terms on solutions to (1). So, for example, if χ is comparatively large, viscous dissipation can hardly have an influence. It affects a flow of air at 20°C more intensely than a flow of water at the same temperature, all other constraints remaining unchanged.

Questions such as 'when can viscous dissipation not be neglected for water? When can thermal diffusion be neglected?' etc., are more easily answered by the use of the formulae in this paper. Let us consider the following example. If water at 20°C with $\Delta T = 1$°C ($\chi = 2.32 \times 10^{28}$) is considered together with $\delta = 10^5$ and $\varepsilon = 10^3$, we have from (7) $\eta = 3.5 \times 10^{-2}$. Multipliers are: $1/\delta = 10^{-5}$; $1/\gamma\delta = 1.43 \times 10^{-6}$; $\varepsilon/\delta^2 = 10^{-7}$; $\eta/\delta = 3.5 \times 10^{-7}$. If reference is made to a

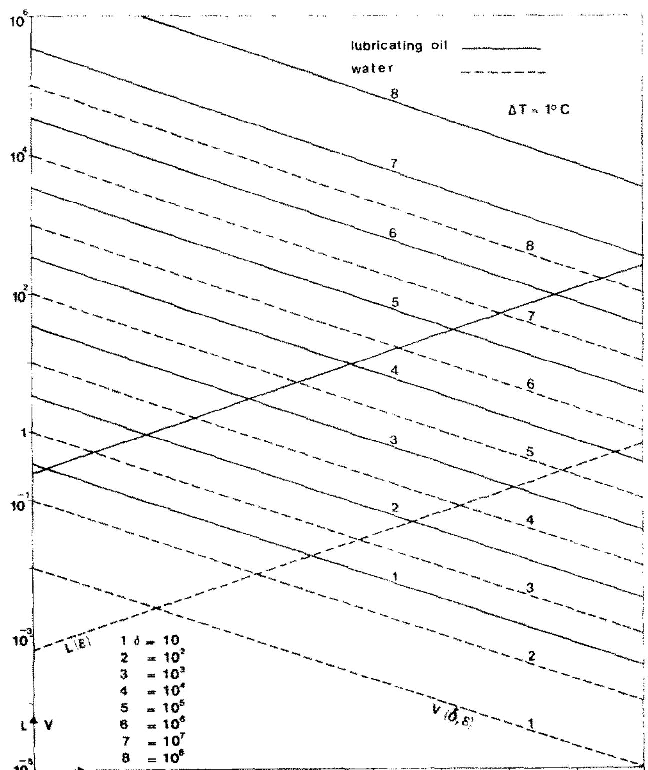

FIG. 3. L in m and V in m s^{-1} as functions of ε for two fluids and several values of δ.

rotating disk with a finite radius, the flow is probably still laminar for $\delta = 10^5$ (this is strictly the case for isothermal conditions). Nothing entitles one to believe that in this case viscous dissipation is *a priori* negligible with respect to other effects, although water is in question.

Another opportunity for exploiting the material of this paper is in the study of a physical system whose configuration is known, but, for reasons of generality, it would be preferable not to prefix its dimensions, operating conditions and working fluid. In this case it is possible to select the most convenient, strictly compatible multipliers for equations (1) before integrating.

A method is offered of localizing the areas of Fig. 1 where the largest or the smallest values of a given coefficient must be sought and what the others are in the quatern. In regards to thermal diffusion, for example, $1/\gamma\delta$ decreases for growing ε and η as is shown by (8), (9) and by Table 2. The influence of this term is reinforced in the lower left region of Fig. 1 and weakened in the upper right one. If it is noted that η/δ increases with η, the points where thermal diffusion may be expected to become negligible against viscous dissipation are within the upper strip of Fig. 1. Some calculations are collected in Table 4 as examples to support this statement and to display the consistent values of multipliers for several common fluids. The calculations of Table 4 and Table 2 also show that ε/δ^2 is lowered faster than $1/\delta$ by an increase in η. Effects linked to these coefficients are probably comparable for values at the bottom of Fig. 1. In the top region viscous forces prevail over natural convection, but they become in turn less and less important with respect to forces of inertia.

If now, restrictions on the amplitude of ΔT are released, the same ratios of dimensionless groups, although multiplied by functions of dimensionless temperature (and possibly pressure), appear in the transport equations with the full variability of all physical properties. As far as compatibility of the four numbers is concerned, it must be observed that the presence, as multipliers, of the dimensionless functions mentioned above (these functions change from one fluid to another, depend on the amplitude of ΔT and on its position in the scale of temperature) prevents any general treatment. The characteristic nature of the fluid plays a role in every case. Arguments in this work still hold with reference to a specified fluid.

3. CONCLUSIONS

A proposal is put forward in this paper to state a compatibility relationship among *Re*, *Gr*, *Ec* without reference to any physical system. This implies the definition of a new dimensionless group, χ, which compactly accounts for the peculiarity of the fluid and ΔT; the latter quantity is seen to be in some way known or evaluable. As a consequence, the multipliers $1/Re$, Gr/Re^2, Ec/Re for the heat and momentum transport equations can be immediately derived. It is shown that more than one value of the multiplier $1/Pr \cdot Re$ of the thermal diffusion term can be consistent with a specified tern of the above mentioned multipliers.

A graphic image as to how *Re*, *Gr*, *Ec* are linked to each other and geometry and operating conditions of a

Table 4. The four multipliers for four η's, two ε's and four fluids at 20°C (from (9))

Fluid		$1/\delta$		$1/\gamma\delta$		ε/δ^2		η/δ	
χ	γ	$\varepsilon = 1$	$\varepsilon = 10^9$	$\varepsilon = 1$	$\varepsilon = 10^9$	$\varepsilon = 1$	$\varepsilon = 10^9$	$\varepsilon = 1$	$\varepsilon = 10^9$
$\eta = 10^8$									
Lub. oil 5.60×10^{20}	1.01×10^4	3.48×10^{-8}	3.48×10^{-11}	3.45×10^{-12}	3.45×10^{-15}	1.21×10^{-15}	1.21×10^{-12}	3.48	3.48×10^{-3}
Water 2.3×10^{28}	7	1.95×10^{-9}	1.95×10^{-12}	2.67×10^{-10}	2.67×10^{-13}	3.5×10^{-18}	3.5×10^{-15}	1.87×10^{-1}	1.87×10^{-4}
Air 1.4×10^{21}	0.7	2.99×10^{-8}	2.99×10^{-11}	4.27×10^{-8}	4.27×10^{-11}	8.94×10^{-16}	8.94×10^{-13}	2.99	2.99×10^{-3}
Mercury 6.37×10^{25}	$2.30\times{}^{-2}$	5.0×10^{-9}	5.0×10^{-12}	2.17×10^{-7}	2.17×10^{-10}	2.5×10^{-17}	2.5×10^{-14}	5.0×10^{-1}	5.0×10^{-4}
$\eta = 10^4$									
Lub. oil		3.48×10^{-6}	3.48×10^{-9}	3.45×10^{-10}	3.45×10^{-13}	1.21×10^{-11}	1.21×10^{-8}	3.48×10^{-2}	3.48×10^{-5}
Water		1.95×10^{-7}	1.95×10^{-10}	2.67×10^{-8}	2.67×10^{-11}	3.5×10^{-14}	3.5×10^{-11}	1.87×10^{-3}	1.87×10^{-6}
Air		2.99×10^{-6}	2.99×10^{-9}	4.27×10^{-6}	4.27×10^{-9}	8.94×10^{-12}	8.94×10^{-9}	2.99×10^{-2}	2.99×10^{-5}
Mercury		5.0×10^{-7}	5.0×10^{-10}	2.17×10^{-5}	2.17×10^{-8}	2.5×10^{-13}	2.5×10^{-10}	5.0×10^{-3}	5.0×10^{-6}
$\eta = 1$									
Lub. oil		3.48×10^{-4}	3.48×10^{-7}	3.45×10^{-8}	3.45×10^{-11}	1.21×10^{-7}	1.21×10^{-4}	3.48×10^{-4}	3.48×10^{-7}
Water		1.95×10^{-5}	1.95×10^{-8}	2.67×10^{-6}	2.67×10^{-9}	3.5×10^{-10}	3.5×10^{-7}	1.87×10^{-5}	1.87×10^{-8}
Air		2.99×10^{-4}	2.99×10^{-7}	4.27×10^{-4}	4.27×10^{-7}	8.94×10^{-8}	8.94×10^{-5}	2.99×10^{-4}	2.99×10^{-7}
Mercury		5.0×10^{-5}	5.0×10^{-8}	2.17×10^{-3}	2.17×10^{-6}	2.5×10^{-9}	2.5×10^{-6}	5.0×10^{-5}	5.0×10^{-8}
$\eta = 10^{-4}$									
Lub. oil		3.48×10^{-2}	3.48×10^{-5}	3.45×10^{-6}	3.45×10^{-9}	1.21×10^{-3}	1.21	3.48×10^{-6}	3.48×10^{-9}
Water		1.95×10^{-3}	1.95×10^{-6}	2.67×10^{-4}	2.67×10^{-7}	3.5×10^{-6}	3.5×10^{-3}	1.87×10^{-7}	1.87×10^{-10}
Air		2.99×10^{-2}	2.99×10^{-5}	4.27×10^{-2}	4.27×10^{-5}	8.94×10^{-4}	8.94×10^{-1}	2.99×10^{-6}	2.99×10^{-9}
Mercury		5.0×10^{-3}	5.0×10^{-6}	2.17×10^{-1}	2.17×10^{-4}	2.5×10^{-5}	2.5×10^{-2}	5.0×10^{-7}	5.0×10^{-10}

physical system is built up. This clarifies the connection between dimensions and operating conditions of the system and the main effects which develop in it.

A result which comes out of the analysis in this work is that natural convection and viscous dissipation cannot simultaneously have an equal and strong influence upon solutions to transport equations in the considered range of Gr.

Finally some examples are given to explain how the material in this work can be conveniently used.

With regard to the second problem, outlined in the Introduction, this paper does not give an exhaustive reply. However, the trend for each multiplier is indicated and it is possible at least to know where each of them tends to become unimportant.

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APPENDIX

Gebhart and Mollendorf write in [1] and [2] that viscous dissipation may be important for very large systems. This statement is consistent with the background supplied by this paper. If the fluid is fixed and ε is kept constant, it stems from (6) and (7) that ΔT can be varied only if L is varied too. This means to consider a sequence of systems, each of them characterized by a couple L, ΔT and by the same fluid. In such a sequence, where ΔT is smaller, L is larger, χ is smaller and, according to (7), viscous dissipation becomes more and more important.

The same result can be reached with a different approach. If the compatibility relationship is derived from (3), we have:

$$\eta = \chi_1 \delta^2 \varepsilon^{-3} \tag{A.1}$$

where

$$\chi_1 = \frac{g\beta L}{c_p}. \tag{A.2}$$

χ_1 is just the dissipation parameter considered by Gebhart. In order to assess the influence of L on η, the change of the former must be isolated. With reference to (A.1) and (A.2), let us keep δ and ε constant and prefix the fluid. Under these hypotheses an increase of L implies a decrease in ΔT and the reasoning proceeds as above. η grows with L, but ΔT is variable too.

It is deduced in this paper that viscous dissipation is more important for systems with small dimensions. So it may seem that this paper contradicts Gebhart and Mollendorf [1] and [2]. This is not the case. While the considerations in (A.1) and (A.2) hold for a variable ΔT and a constant ε. Figure 3 in this paper displays the course of the function $L(\varepsilon)$ for a strictly constant ΔT. Clearly this is a different point of view.

COMPATIBILITE ENTRE LES GROUPES ADIMENSIONNELS DES EQUATIONS AUX DERIVEES PARTIELLES DE NAVIER–STOKES ET DE L'ENERGIE

Résumé—On propose une méthode pour obtenir une relation de compatibilité entre le nombre de Prandtl, le nombre de Reynolds, celui de Grashof et celui de Eckert qui sont présents dans les équations différentielles de transfert de chaleur et de quantité de mouvement. On a ainsi la possibilité de calculer des multiplicateurs cohérents pour tous les termes des susdites équations sans se référer à aucun système physique. L'analyse a détecté que la convection naturelle et la dissipation visqueuse ne peuvent pas simultanément influencer les solutions des équations de transfert avec une égale et haute intensité.

VEREINBARKEIT UND WECHSELWIRKUNG ZWISCHEN DEN KENNGRÖßEN DER NAVIER–STOKES- UND DER ENERGIE-DIFFERENTIALGLEICHUNG

Zusammenfassung—Es wird eine Methode mitgeteilt, die eine Beziehung über die Vereinbarkeit von Werten der Prandtl-, Reynolds-, Grashof- und Eckert-Zahlen in den Differentialgleichungen für den Wärme- und Impulstransport liefert. Damit kann man konsistente Multiplikatoren für jeden Term der genannten Gleichungen ohne Bezug auf ein bestimmtes System berechnen. Es zeigt sich, daß freie Konvektion und viskose Dissipation nicht gleichzeitig die Lösung einer Transportgleichung mit gleicher und starker Intensität beeinflussen können.

СОВМЕСТИМОСТЬ И ВЗАИМОСВЯЗЬ МЕЖДУ БЕЗРАЗМЕРНЫМИ КОМПЛЕКСАМИ В ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЯХ НАВЬЕ–СТОКСА И ЭНЕРГИИ

Аннотация — Предложен метод получения условий совместимости чисел Прандтля, Рейнольдса, Грасгофа и Эккерта, входящих в дифференциальные уравнения переноса тепла и импульса. Он позволяет рассчитывать согласующие множители для всех членов названных уравнений без апелляции к конкретной физической системе. Анализ показывает, что естественная конвекция и вязкая диссипация не могут одновременно оказывать одинаково большое влияние на решение уравнений переноса.